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An approximation algorithm with performance guarantees for the maximum traveling salesman problem on special matrices

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Abstract

In this paper, we investigate the maximum traveling salesman problem (Max-TSP) on quasi-banded matrices. A matrix is quasi-banded with multiplier α if all its elements contained within a band of several diagonals above and below the principal diagonal are non-zero, and any element in the band is at least α times larger than the maximal element outside the band. We investigate the properties of the Max-TSP on the quasi-banded matrices, prove that it is strongly NP-hard and derive a linear-time approximation algorithm with a guaranteed performance. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the maximum traveling salesman problem (Max-TSP), the input consists of a set N of n *cities* together with a $n \times n$ matrix W of non-negative real numbers $w(i,j)$ defined for every pair of distinct cities $i,j \in N$. The goal is to find a cyclic permutation, or *tour*, of the cities, $\pi \in \boldsymbol{C}_n$, that maximizes the total tour length. The *length* of tour $\pi = \{(i,\pi(i)), i=1,\ldots,n\}$ is $L(W,\pi) = \sum_{i=1}^{n} w(i,\pi(i))$; $\boldsymbol{C}_n$ is the set of all cyclic permutations. The problem is known to be a model for scheduling a single processor with setups arising in manufacturing, computing, VLSI design and many other applications [12]. The Max-TSP is known to be strongly NP-hard [8]. Polynomial-time algorithms (exact or approximate with performance guarantees) for special cases of the problem have been suggested in [1,2,4–7,9–11,13].

In this paper, we investigate the Max-TSP on non-negative quasi-banded matrices. A matrix $W = [w(i,j)]$ is called (p,q)*-quasi-banded with multiplier* α if: (1) all its

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elements contained within a band of p diagonals above the principal diagonal and q diagonals below it are non-zero, and (2) $\min_{j-i\in(-q,\dots,p)} w(i,j) \geqslant \alpha \max_{j-i\notin(-q,\dots,p)} w(i,j)$, where $\alpha > 1$. The study of the traveling salesman problem on banded matrices was pioneered in [4] where the Max-TSP on the simplest symmetrical banded matrices, (that is, when $p=q=1$) was investigated. In this paper, as a next step, we study the Max-TSP on the simplest *asymmetrical* banded matrices, that is, when $p=2$ and $q=0$. The asymmetric structure of the banded matrices has required another approach to the problem analysis than that in the symmetric case and led to another accuracy estimation for the approximation algorithm. Notice that the elements of the main diagonal of W are not required for the statement of the traveling salesman problem, and, hence, throughout the paper they will not be examined. This implies, in particular, that our results will also be valid for the matrices in which the elements of the main diagonal are zeros, and $\min_{j-i\in(-q,\dots,p)\setminus\{0\}} w(i,j) \geqslant \alpha \max_{j-i\notin(-q,\dots,p)} w(i,j)$, where $\alpha > 1$.

In Section 2, the strong NP-hardness of the Max-TSP on the quasi-banded matrices is established. In Section 3, an exact linear-time algorithm for the Max-TSP on (2,0)-banded matrices is derived. In Section 4, the algorithm derived for the banded matrices is used to obtain an approximate solution with performance guarantees for the quasi-banded matrices.

2. Max-TSP on quasi-banded matrices is strongly NP-hard

For any $n \times n$ matrix $D=[d(i,j)]$, we can construct the weighted, complete digraph $G(D)$ with vertices $1,\dots,n$ such that the length of arc (i,j) equals $d(i,j)$. For any tour $\pi \in \boldsymbol{C}_n$, there is a corresponding Hamiltonian cycle H in $G(D)$, i.e., a simple directed cycle containing all vertices of $G(D)$. The *length* $L(H)$ of cycle H is the sum of the lengths of its arcs.

Theorem 1. *The Max-TSP for the non-negative* (2,0)*-quasi-banded matrices with* $\alpha > 1$ *is strongly* NP*-hard.*

Proof. Consider an arbitrary instance of the Max-TSP on $n+1$ cities with a $(n+1)\times(n+1)$ non-negative matrix $D=[d(i,j)$: $i,j=0,1,\dots,n,\ i\neq j]$ as the input. Let $G(D)$ be its corresponding digraph.

We shall construct an equivalent instance of the Max-TSP on a (2,0)-quasi-banded matrix with multiplier $\alpha > 1$, having the same computational complexity. For this purpose, denote $A=3\alpha n \max_{i,j} d(i,j), B=\alpha \max_{i,j} d(i,j)$, and define a $3n\times 3n$ matrix $W=[w(i,j)]$ as follows:

$$w(3i-2,3i)=A \quad \text{for } 1\leqslant i\leqslant n,$$

$$w(3i,3i+1)=A \quad \text{for } 1\leqslant i\leqslant n-1,$$

$$w(3n,3i-1)=d(0,i) \quad \text{for } 1\leqslant i\leqslant n,$$

$$
\begin{array}{ll}
w(3i-1,1)=d(i,0) & \text{for } 1 \leqslant i \leqslant n,\\
w(3i-1,3j-1)=d(i,j) & \text{for } l \leqslant i,j \leqslant n,\\
w(i,j)=B & \text{for } j-i \in \{1,2\}, w(i,j) \neq A,\\
w(i,j)=0 & \text{for all other } i,j.
\end{array}
$$

Notice that here the elements of the main diagonal are zeros; this is unessential for our purposes because the elements are not required in the statement of the Max-TSP. The obtained matrix W, with the exception of the main diagonal, is non-negative and quasi-banded with $\alpha > 1$. Now we will show that the optimal solution values to the Max-TSP on W and D differ by $(2n-1)A$.

Evidently, any maximum tour in $G(W)$ must contain all the arcs of length A. Remove from $G(W)$ the path

$$
\begin{aligned}
&(1 \to 3 \to 4 \to 6 \to \cdots \to 3i-3 \to 3i-2 \to 3i \to \cdots\\
&\quad \to 3n-3 \to 3n-2 \to 3n),
\end{aligned}
$$

all arcs which are of length A. Remove, also, the arcs leaving and entering the inner nodes of this path as well as the arcs leaving node 1 and entering node $3n$. Consider nodes 1 and $3n$ as a single node and denote it by $(1\&3n)$. As a result, we obtain a new digraph $G'(D)$, corresponding to the original digraph $G(D)$, such that node $(1\&3n)$ of $G'(D)$ corresponds to node 0 of $G(D)$, and node $3i-1$ of $G'(D)$ corresponds to node i of $G(D)$, $i=1,\dots,n$.

In order to solve the Max-TSP on an arbitrary non-negative $(n+1)\times(n+1)$-matrix D, it is sufficient to construct a corresponding $(2,0)$-quasi-banded $3n \times 3n$-matrix W the elements of which are at most $3n\alpha$ times larger than the elements of D, and to solve to optimality the Max-TSP on the W. (Notice that the matrix W is constructed from D in linear time). Recall that the Max-TSP on D is strongly NP-hard (see [8]). Therefore, the Max-TSP on W is strongly NP-hard as well, which proves the theorem. $\square$

3. Analysis of the Max-TSP on banded matrices

In this section, we study some properties of banded matrices which are useful for solving the Max-TSP on quasi-banded matrices. A matrix is called (p,q)-*banded* if all its non-zero elements are contained within a band of p diagonals above the principal diagonal and q diagonals below it.

We start with studying the sets $\boldsymbol{M}_{(2,0)}(n)$ of all the $n\times n$, non-negative $(2,0)$-banded matrices. We investigate the minimal set of permutations (tours) which contains at least one optimal tour for every instance of the Max-TSP on the matrices from $\boldsymbol{M}_{(2,0)}(n)$.

Definition (Blokh [3]). Given a set of matrices $\boldsymbol{M} \subseteq \boldsymbol{M}_n$ (where $\boldsymbol{M}_n$ is the set of all $n \times n$ matrices), the permutation set $\boldsymbol{K} \subseteq \boldsymbol{C}_n$ is called the *minimal set* for $\boldsymbol{M}$, if the following conditions hold:

(1) for any matrix W from $\boldsymbol{M}$, at least one of its longest tours belongs to $\boldsymbol{K}$, and
(2) any proper subset $\boldsymbol{K}' \subset \boldsymbol{K}$ does not satisfy condition (1).

Denote by $\boldsymbol{M}'_{(2,0)}(n)$ the subset of matrices from $\boldsymbol{M}_{(2,0)}(n)$ which have $w(1,2)=0$. We need the following lemmas:

Lemma 1. *If a matrix $W \in \boldsymbol{M}_{(2,0)}(n)$ then there exists a tour τ, maximal for W, which includes either the entry $w(1,2)$ or the entry $w(1,3)$.*

Lemma 2. *If a matrix $W \in \boldsymbol{M}'_{(2,0)}(n)$ then there exists a tour τ, maximal for W, which either includes both the entries $w(1,2)$ and $w(2,3)$, or includes the entry $w(1,3)$.*

The proofs are given in the appendix.

We can now describe an iterative algorithmic scheme which generates the minimal set of tours for the set of matrices $\boldsymbol{M}_{(2,0)}(n)$. Its kth iteration ($k=n,n-1,\ldots,3$) is as follows:

Scheme S

Stage 1. $\boldsymbol{M}_{(2,0)}(k)$: $w(1,2) \to \boldsymbol{M}_{(2,0)}(k-1)$; $\boldsymbol{M}_{(2,0)}(k)$: $w(1,3), T \to \boldsymbol{M}'_{(2,0)}(k-1)$.
Stage 2. $\boldsymbol{M}'_{(2,0)}(k)$: $w(1,2), w(2,3) \to \boldsymbol{M}_{(2,0)}(k-2)$; $\boldsymbol{M}'_{(2,0)}(k)$: $w(1,3), T \to \boldsymbol{M}'_{(2,0)}(k-1)$.

The symbolic expression $\boldsymbol{M}_{(2,0)}(k)$: $w(1,2) \to \boldsymbol{M}_{(2,0)}(k-1)$ means that if a matrix $\boldsymbol{M}_{(2,0)}(k)$ is obtained in the process of constructing a tour then:
(1) the entry $w(1,2)$ is included into the tour,
(2) the row 1 and the column 2 are deleted, and
(3) the process is repeated for the obtained matrix $\boldsymbol{M}_{(2,0)}(k-1)$ of a smaller size.
The expression $\boldsymbol{M}_{(2,0)}(k)$: $w(1,3), T \to \boldsymbol{M}'_{(2,0)}(k-1)$ describes another alternative for a matrix $\boldsymbol{M}_{(2,0)}(k)$ obtained in the process of constructing a tour:
(1) the entry $w(1,3)$ is included into the tour,
(2) the row 1 and the column 3 are deleted,
(3) the columns 1 and 2 in the obtained matrix are transposed,
(4) the process is repeated for the obtained matrix $\boldsymbol{M}'_{(2,0)}(k-1)$ of a smaller size.
The meaning of symbolic expressions at Stage 2 of the above scheme is similar.

Definition. The elements $w(i,j)$ of W for which $j-i \in \{1,2\}$ are called *basic*; all other elements are called *non-basic*.

Let $\boldsymbol{K}_{(2,0)}(n)$ denote the set of all possible tours which can be obtained as a result of applying the above scheme $\boldsymbol{S}$ to the matrices from $\boldsymbol{M}'_{(2,0)}(n)$. The following theorem motivates the introduction of the above algorithmic scheme.

Theorem 2. *The set $\boldsymbol{K}_{(2,0)}(n)$ is a minimal set of tours for $\boldsymbol{M}_{(2,0)}(n)$.*

The proof is given in the appendix.

Remark. It is possible to prove (see [5]) that all the minimal sets for $\boldsymbol{M}_{(2,0)}(n)$ have the same number of elements, and this number is $O(\lambda^n)$ where $1.7548 < \lambda < 1.7549$.

Observation. Let $W=[w(i,j)]\in \boldsymbol{M}_{(2,0)}(n)$ be a given matrix, and $V=[v(i,j)]$ the matrix obtained at a certain iteration of Scheme $\boldsymbol{S}$. If $V\in \boldsymbol{M}_{(2,0)}(k)$ then the elements $v(1,2)$ and $v(1,3)$ are basic elements of W. If $V\in \boldsymbol{M}'_{(2,0)}(k)$ then the elements $v(1,3)$ and $v(2,3)$ are basic elements whereas $v(1,2)$ is a non-basic element of the W.

The claim follows straightforwardly from the construction of Scheme $\boldsymbol{S}$.

Theorem 3. *Any tour from* $\boldsymbol{K}_{(2,0)}(n)$ *contains at most* $[n/3]+1$ *non-basic elements.*

Proof. A non-basic element may enter a tour if and only if a matrix V is from $\boldsymbol{M}'_{(2,0)}(k)$ and this non-basic element is to be $v(1,2)$. The element $v(2,3)$ of V from $\boldsymbol{M}'_{(2,0)}(k)$ must be in this tour. Further, the element $v(1,3)$ of the matrix from $\boldsymbol{M}_{(2,0)}(k+1)$ obtained at the previous iteration of $\boldsymbol{S}$ also must be in the tour. But the two latter elements are basic in the original matrix W. Therefore, the number of non-basic elements in any tour is at most $[n/3]+1$. $\square$

4. An exact algorithm for the Max-TSP on (2,0)-banded matrices

In this section, we consider an exact linear-time algorithm for solving the Max-TSP on the set $\boldsymbol{M}_{(2,0)}(n)$ of all $(2,0)$-banded, non-negative $n\times n$-matrices W.

Let us denote $f_1(n)=\max_\pi L(W,\pi)=\max_\pi \sum_{i=1}^n w(i,\pi(i))$.

Theorem 4. *An optimum solution to the Max-TSP on any* $(2,0)$*-banded matrix can be found in linear time by using the following recursive equations*:

$$\begin{aligned} f_1(i)=\max\{&w(n-i+1,n-i+2)+f_1(i-1),\\ &w(n-i+1,n-i+3)+f_2(i-1)\}, \end{aligned}\tag{1}$$

$$f_2(i)=\max\{f_1(i-1),w(n-i+1,n-i+3)+f_2(i-1)\}$$

with the starting conditions

$$f_1(2)=w(n-1,n),f_2(2)=0.$$

The proof is given in the appendix.

Obviously, the complexity of running the recursive formulas above is $O(n)$. Any optimal solution provided by the algorithm in Theorem 4 corresponds to an optimal solution from the set $\boldsymbol{K}_{(2,0)}(n)$ obtained by using Scheme $\boldsymbol{S}$.

5. Approximation algorithm for Max-TSP on quasi-banded matrices

In this section, the exact linear-time algorithm for the Max-TSP on banded matrices is used to yield approximate solutions with performance guarantees for the problem on the quasi-banded matrices.

Lemma 3. *If $W=[w(i,j)]$ is a $(2,0)$-quasi-banded matrix with $\alpha>3$ then there exists an optimal tour for W containing at most $[n/3]+1$ non-basic elements (where $[x]$ denotes the integral part of x).*

Proof. For each tour π in any matrix W, there is a corresponding *trajectory* $t=\{w(\pi(1),\pi(2)),\ldots,w(\pi(2),\pi(3)),\ldots,w(\pi(n),\pi(1))\}$. In what follows, we will identify tour π and trajectory t. Suppose that an optimal tour π for $W=[w(i,j)]$ contains more than $[n/3]+1$ non-basic elements. Consider the matrix $W'=[w'(i,j)]\in \boldsymbol{M}_{(2,0)}(n)$ such that:

$$w'(i,j)=1 \quad \text{if } (i,j)\in\pi \text{ and } j-i\in\{1,2\},$$
$$w'(i,j)=0 \quad \text{otherwise.}$$

Let π' be an optimal tour for W and $\pi'\in \boldsymbol{K}_{(2,0)}(n)$. According to Theorem 3, the tour π' contains at most $[n/3]+1$ non-basic elements. The tour π' contains all the basic elements of the tour π. According to our assumption, the tour π contains less than $n-[n/3]-1$ basic elements. At the same time, the tour π' contains at least $n-[n/3]-1$ basic elements, and π' contains all basic elements of π. Therefore, π' contains all the basic elements of π and at least one basic element more. Denote this "additional" basic element by $w(u,z)$.

We can show that the element $w(u,z)$ may be introduced into the tour π by deleting not more than three non-basic elements of π. Let l (respectively, k) be a node immediately following (respectively, preceding) node u in π, and p (respectively, q) a node immediately preceding (respectively, following) z in π. The elements $w(u,l)$ and $w(p,z)$ are non-basic in π. Let us connect nodes u and z, and consider the cycle $(u,z,q,\ldots,k)$. Except for $w(u,z)$, all its elements belong to π. Clearly, this cycle contains a non-basic element (as any other cycle of W from $\boldsymbol{M}_{(2,0)}(n)$ does). Assume that this element is $w(s,t)$. Consider the tour $\tau=(\pi\setminus\{w(u,l),w(p,z),w(s,t)\})\cup\{w(u,z),w(p,t),w(s,l)\})$. It contains the basic element $w(u,z)$ (and it does not have three non-basic elements of π). Since $\alpha>3$, the element $w(u,z)\geqslant w(u,z)+w(p,t)+w(s,l)$. By our assumption, tour π is optimal. Then $\sum_{i=1}^{n} w(i,\tau(i))=\sum_{i=1}^{n} w(i,\pi(i))$. $\square$

Theorem 5. *Let $W=[w(i,j)]$ be a $(2,0)$-quasi-banded $n\times n$-matrix with $\alpha\geqslant 3$, and the matrix $W'=[w'(i,j)]\in \boldsymbol{M}_{(2,0)}(n)$ be such that:*

$$w'(i,j)=w(i,j) \; if \; j-i\in\{1,2\} \quad \textit{and}$$
$$w'(i,j)=0 \quad \textit{otherwise.}$$

Denote by L^* (*respectively*, L'^*) *the optimum of the Max-TSP for* W (*respectively*, *for* W'). *Then* $(L^* - L'^*)/L^* \leqslant (n+3)/3\alpha(n-1)$.

Proof. Let π be an optimal tour of the TSP for W containing at most $[n/3]+1$ non-basic elements. Denote the sum of all non-basic elements of π by L_1, and set $L_2 := L^* - L_1$.

Notice that $L_2 = \sum_{i=1}^{n} w'(i,\pi(i))$, and $L'^* \geqslant L_2$.

Since $w(i,j) \geqslant w'(i,j)$, for all i and j, we have: $L^* \geqslant L'^*$.

Therefore,

$$L_2 \leqslant L'^* \leqslant L^*, \text{and } (L^* - L'^*)/L^* \leqslant (L^* - L_2)/L^* = L_1/L^*.$$

Since L_1 is the sum of at most $[n/3]+1$ non-basic elements, we have that

$$L_1 \leqslant ([n/3]+1) \max_{j-i \notin \{1,2\}} w(i,j).$$

There is a tour containing all $n-1$ elements of the diagonal d lying above the principal diagonal together with an element outside the considered diagonal d. Obviously, L^* is no less than the length of the considered tour, so $L^* \geqslant (n-1)\min_{j-i \in \{i,2\}} w(i,j)$.

Then

$$\begin{aligned} L_1/L^* &\leqslant [([n/3]+1) \max_{j-i \notin \{1,2\}} w(i,j)/(n-1) \min_{j-i \in \{1,2\}} w(i,j) \\ &\leqslant (n+3)/3\alpha(n-1). \quad \square \end{aligned}$$

6. Concluding remarks

The study of the traveling salesman problem on banded matrices has been started in [4] where the Max-TSP on the simplest symmetrical banded matrices has been investigated. In this paper, as a next step, we study the Max-TSP on the simplest asymmetrical banded matrices. This problem can be interpreted as an asymmetrical version of the transportation model [4] in which "cities" have been located along a highway, and the traveling salesman gained a higher profit if he passed a shorter distance; as opposed to the symmetric two-way main road in [4], our model treats the case of a one-way highway.

Our analysis of the asymmetrical case has been based on the concept of the minimal set of tours introduced in [3]. This approach, unlike the graph approach for the symmetrical case in [4], has taken the matrix asymmetry into account. The guaranteed accuracy of the approximation algorithm for the $(2,0)$-quasi-banded matrices equal to $(n+3)/3\alpha(n-1)$, is different from the performance ratio for the $(1,1)$-quasi-banded matrices in [4].

The approach suggested in this paper can be extended to the Maximum-TSP on other matrices. The next step in this direction is to exactly solve the Maximum-TSP on the (3,0)- and (2,1)-banded matrices. Another possible direction for future research is to design approximation algorithms with performance guarantees for the Maximum-TSP on other quasi-banded and more general matrices.

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Appendix A.

A.1. The proof of Lemma 1

Assume that some optimal tour $\pi \in \boldsymbol{C}_n$ contains neither (1,2), nor (1,3). The following tour $\tau = (\pi \setminus \{(k,1),(1,l),(m,2)\}) \cup \{(m,1),(1,2),(k,l)\}$ will be optimal. Indeed, $w(k,1) = w(1,l) = 0$ $(l \neq 2,3)$ and $w(m,2) = 0$ $(m \neq 1)$. Then $L(\tau) \geqslant L(\pi)$, that is, τ is optimal while $(1,2) \in \tau$.

A.2. The proof of Lemma 2

Let π be an optimal tour containing one of two entries, (1,2) or (1,3). Existence of such a tour follows from Lemma 1. Suppose that π contains (1,2), and does not contain (1,3) and (2,3) (else we have the required result). Define a tour τ as follows: $\tau \equiv (\pi \setminus \{(k,1),(1,2),(r,3)\}) \cup \{(1,3),(k,2),(r,1)\})$. Then τ is an optimal tour of the desired type. This proves the Lemma.

A.3. The proof of Theorem 2

Recall that an element $w(i,j)$ from W is called a *base* if $j - i$ equals 1 or 2. We wish to show that the set $\boldsymbol{K}_{(2,0)}(n)$ satisfies conditions (1)–(2) in the definition of the minimal set.

First, we prove the following claim. Any tour from $\boldsymbol{K}_{(2,0)}(n)$ is determined by its basic elements. This means that, given the basic elements of a tour in $\boldsymbol{K}_{(2,0)}(n)$, the all non-basic elements of the tour can also be found.

Indeed, the above algorithm provides that a zero-valued non-basic element can be included into an optimal route in $\boldsymbol{K}_{(2,0)}(n)$ if and only if the algorithm, at some step, received a matrix from $\boldsymbol{M}'_{(2,0)}(k)$ and this zero element is (1, 2).

Let W be from $\boldsymbol{M}_{(2,0)}(n)$; π from $\boldsymbol{K}_{(2,0)}(n)$; $w(1,\pi(1)), w(2,\pi(2)), \ldots, w(s,\pi(s))$ are the basic elements of W, while $w(s+1,\pi(s+1))$ is non-basic. After deleting rows $1,2,\ldots,s$ and columns $\pi(1),\pi(2),\ldots,\pi(s)$, we obtain the matrix $V = [v(i,j)]$ from

$M'_{(2,0)}(n-s)$. In this matrix, the element $w(s+2,\pi(s+2))=v(2,3)$. This implies that $w(s+1,\pi(s+1))=v(1,2)$, that is, the element $w(s+1,\pi(s+1))$ can be determined.

The set $\boldsymbol{K}_{(2,0)}(n)$, evidently, satisfies Condition (1) in the definition of the minimal set of tours. Let us prove that (2) also holds. Assume that there exists a sub-set $\boldsymbol{K}(n) \subset \boldsymbol{K}_{(2,0)}(n)$ satisfying (1). We can show that then $\boldsymbol{K}(n)=\boldsymbol{K}_{(2,0)}(n)$ (a contradiction).

Let a tour $\tau \in \boldsymbol{K}_{(2,0)}(n)$. Consider a matrix $W \in \boldsymbol{M}_{(2,0)}(n)$ such that $w(i,j)=1$ if the element (i,j) is basic and belongs to τ; otherwise $w(i,j)=0$. The trajectory τ is maximal for the matrix $W=[w(i,j)]$. There exists a trajectory $\tau' \in \boldsymbol{K}(n)$ which is maximal for the matrix $W=[w(i,j)]$. All elements of W equal to 1 must be included into the trajectory τ' (otherwise, it will not be maximal). Hence, the basic elements in tours τ and τ' coincide. But $\tau' \in \boldsymbol{K}(n) \subset \boldsymbol{K}_{(2,0)}(n)$, and trajectories in $\boldsymbol{K}_{(2,0)}(n)$ are determined by their basic elements. Therefore, $\tau=\tau'$ and $\boldsymbol{K}_{(2,0)}(n) \subset \boldsymbol{K}(n)$. This implies that $\boldsymbol{K}(n)=\boldsymbol{K}_{(2,0)}(n)$.

A.4. The proof of Theorem 4

We shall prove the result by mathematical induction in the matrix size n. As a basis of induction, we take $n=4$. Consider the following set $\boldsymbol{K} \subset \boldsymbol{C}_4$:

$$\boldsymbol{K}=\{(1,2,3,4),(1,2,4,3),(1,3,2,4),(1,3,4,2)\}. \tag{A.1}$$

By direct considerations, one can see that this set contains all possible optimal tours for the set $\boldsymbol{M}_{(2,0)}(4)$. Further, it is not difficult to see that if a tour from $\boldsymbol{K}$ is optimal, for some matrix from $\boldsymbol{M}_{(2,0)}(4)$, then it can be obtained using the recursive formulas (1). This can be easily done straightforwardly. Indeed, the starting conditions from Theorem 4 yield:

$$f_1(2)=w(3,4), f_2(2)=0 \text{ (for } i=2).$$

For $i=3$, Eqs. (1) of Theorem 4 provide:

$$f_1(3)=\max\{w(2,3)+f_1(2),w(2,4)+f_2(2)\}=\max\{w(2,3)+w(3,4),w(2,4)\},$$

$$f_2(3)=\max\{f_1(2),w(2,4)+f_2(2)\}=\max\{w(3,4),w(2,4)\}.$$

For $i=4$, Eqs. (1) of Theorem 4 provide:

$$\begin{aligned} f_1(4)&=\max\{w(1,2)+f_1(3),w(1,3)+f_2(3)\}\\ &=\max\{w(1,2)+\max\{w(2,3)+w(3,4),w(2,4)\},\\ &\quad w(1,3)+\max\{w(3,4),w(2,4)\}\}\\ &=\max\{w(1,2)+w(2,3)+w(3,4),w(1,2)+w(2,4)\},\\ &\quad w(1,3)+w(3,4),w(1,3)+w(2,4). \end{aligned}$$

Thus, if a tour from $\boldsymbol{K}$ in (2) is optimal then the recursive formulas (1) will recognize it.

Assume now that the theorem is valid for all $k \leqslant n-1$, and let us show that it is true for $k=n$. Let $W \in \boldsymbol{M}_{(2,0)}(n)$, and consider the first formula in (1) for $i=n$:

$$f_1(n)=\max\{w(1,2)+f_1(n-l), w(1,3)+f_2(n-1)\}.$$

According to Lemma 1, the optimal tour of the salesman for W either contains entry (1,2), or entry (1,3). Assume that the optimal tour for W contains entry (1,2). Delete in $W \in \boldsymbol{M}_{(2,0)}(n)$ row 1 and column 2. We obtain a matrix from $W \in \boldsymbol{M}_{(2,0)}(n-1)$, and we can, by the induction hypothesis, apply formulas (1) to it. Then we obtain an optimal tour starting in the second city (corresponding to the matrix $W \in \boldsymbol{M}_{(2,0)}(n-1)$) and terminating in the first city. By adding to this tour the link (1,2), we will have the optimal tour of the TSP for the $W \in \boldsymbol{M}_{(2,0)}(n)$.

Similar arguments are valid when the optimal tour for $W \in \boldsymbol{M}_{(2,0)}(n)$ contains the entry (1,3). In this case, delete in W row 1 and column 3, and in the obtained matrix interchange columns 1 and 2. As a result, we have a special case of the matrix from $W \in \boldsymbol{M}_{(2,0)}(n-1)$. By the induction hypothesis, we can apply again formulas (1) to it, and obtain an optimal tour starting in 3 and terminating in 1. Then we add the link (1,3) and obtain the optimal tour for $W \in \boldsymbol{M}_{(2,0)}(n)$. $\square$

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